Name (IN CAPITALS): **Version #1**

Instructor: Dora The Explorer

Math 10560 Exam 1
Feb. 18, 2025.

- The Honor Code is in effect for this examination. All work is to be your own.
- Please turn off (and Put Away) all cellphones, smartwatches and electronic devices.
- Calculators are **not** allowed.
- The exam lasts for 1 hour and 15 minutes.
- Be sure that your name and your instructor's name are on the front page of your exam.
- Be sure that you have all 13 pages of the test.
- Each multiple choice question is worth 7 points. Your score will be the sum of the best 10 scores on the multiple choice questions plus your score on questions 13-16.

PLEASE MARK YOUR ANSWERS WITH AN X, not a circle!

1.	(●)	(b)	(c)	(d)	(e)
2.	(●)	(b)	(c)	(d)	(e)
3.	(●)	(b)	(c)	(d)	(e)
4.	(●)	(b)	(c)	(d)	(e)
5.	(●)	(b)	(c)	(d)	(e)
6.	(●)	(b)	(c)	(d)	(e)
7.	(●)	(b)	(c)	(d)	(e)
8.	(●)	(b)	(c)	(d)	(e)
9.	(●)	(b)	(c)	(d)	(e)
10.	(●)	(b)	(c)	(d)	(e)
11.	(●)	(b)	(c)	(d)	(e)
12.	(●)	(b)	(c)	(d)	(e)

Please do NOT write in this box.

Multiple Choice	
13.	
14.	
15.	
16.	
Total	

Name (IN CAPITALS): ____________________________

Instructor: ________________________

Math 10560 Exam 1
Feb. 18, 2025.

- The Honor Code is in effect for this examination. All work is to be your own.
- Please turn off (and Put Away) all cellphones, smartwatches and electronic devices.
- Calculators are **not** allowed.
- The exam lasts for 1 hour and 15 minutes.
- Be sure that your name and your instructor's name are on the front page of your exam.
- Be sure that you have all 13 pages of the test.
- Each multiple choice question is worth 7 points. Your score will be the sum of the best 10 scores on the multiple choice questions plus your score on questions 13-16.

PLEASE MARK YOUR ANSWERS WITH AN X, not a circle!

1.	(a)	(b)	(c)	(d)	(e)
2	(a)	(b)	(c)	(d)	(e)
3.	(a)	(b)	(c)	(d)	(e)
4.	(a)	(b)	(c)	(d)	(e)
5.	(a)	(b)	(c)	(d)	(e)
6.	(a)	(b)	(c)	(d)	(e)
7.	(a)	(b)	(c)	(d)	(e)
8.	(a)	(b)	(c)	(d)	(e)
9.	(a)	(b)	(c)	(d)	(e)
10.	(a)	(b)	(c)	(d)	(e)
11.	(a)	(b)	(c)	(d)	(e)
12.	(a)	(b)	(c)	(d)	(e)

Please do NOT write in this box.

Multiple Choice	________
13.	________
14.	________
15.	________
16.	________
Total	________

 Initials: ____________

Multiple Choice

1.(7pts) The function

$$f(x) = x^3 + x + e^x + 2$$

is a one-to-one function (there is no need to check this). What is $(f^{-1})'(3)$?

Solution: (a) $\frac{1}{2}$. We know that $(f^{-1})'(3) = \dfrac{1}{f'[f^{-1}(3)]}$. Firstly, we need to find $f^{-1}(3)$. Since $f(f^{-1}(3)) = 3$, we know $f^{-1}(3)^3 + f^{-1}(3) + e^{f^{-1}(3)} + 2 = 3$. Solving this equation, we see $f^{-1}(3) = 0$. Secondly, we find $f'(x) = 3x^2 + 1 + e^x$, so $f(f^{-1}(3)) = f'(0) = 2$. Substitute this back to the formula, we learn that $(f^{-1})'(3) = \dfrac{1}{f'[f^{-1}(3)} = \dfrac{1}{2}$.

(a) $\dfrac{1}{2}$ (b) $28 + e^3$ (c) 2 (d) $\dfrac{1}{28 + e^3}$ (e) $\dfrac{1}{4 + e}$

2.(7pts) Find $\dfrac{dy}{dx}$ if

$$y = \frac{(x^4+1)(x+2)^{50}}{\sqrt{x^2+4}}.$$

(a) $\dfrac{(x^4+1)(x+2)^{50}}{\sqrt{x^2+4}}\left(\dfrac{4x^3}{x^4+1} + \dfrac{50}{x+2} - \dfrac{x}{x^2+4}\right)$

Take ln on both sides, we get $\ln(y) = \ln(x^4+1) + 50\ln(x+2) - \frac{1}{2}\ln(x^2+4)$. Then take derivaitive on both sides, it gives $\frac{1}{y}y' = \frac{4x^3}{x^4+1} + \frac{50}{x+2} - \frac{x}{x^2+4}$. By multiplying y on both sides, we get the answer.

(a) $\dfrac{(x^4+1)(x+2)^{50}}{\sqrt{x^2+4}}\left(\dfrac{4x^3}{x^4+1} + \dfrac{50}{x+2} - \dfrac{x}{x^2+4}\right)$

(b) $\dfrac{4x^3}{x^4+1} + \dfrac{50}{x+2} - \dfrac{x}{x^2+4}$

(c) $\dfrac{\left(\dfrac{4x^3}{x^4+1}\right)\left(\dfrac{50}{x+2}\right)}{\dfrac{x}{x^2+4}}$

(d) $\dfrac{(x^4+1)(x+2)^{50}}{\sqrt{x^2+4}}\left(\dfrac{4x^3}{x^4+1} + \dfrac{50}{x+2} + \dfrac{2x}{x^2+4}\right)$

(e) $\left(\dfrac{(x^4+1)(x+2)^{50}}{\sqrt{x^2+4}}\right)\dfrac{\left(\dfrac{4x^3}{x^4+1}\right)\left(\dfrac{50}{x+2}\right)}{\left(\dfrac{x}{x^2+4}\right)}$

3.(7pts) Evaluate the indefinite integral:

$$\int \frac{dx}{x(\ln x)^2}.$$

(a) $\dfrac{-1}{\ln(x)} + C$

Take $u = \ln x$ so that $du = \frac{1}{x}dx$. Hence, the indefinite integral is

$$\int \frac{dx}{x(\ln x)^2} = \int \frac{1}{u^2} du = -\frac{1}{u} + C = -\frac{1}{\ln x} + C$$

.

(a) $\dfrac{-1}{\ln(x)} + C$

(b) $\ln\left(x(\ln x)^2\right) + C$

(c) $\dfrac{1}{x\ln(x)} + C$

(d) $\dfrac{-3}{(\ln(x))^3} + C$

(e) $\ln\left((\ln x)^2\right) + C$

4.(7pts) Compute $f'(x)$ if

$$f(x) = 3^{x^2+1}.$$

(a) $2x(\ln 3)3^{x^2+1}$

Take ln on both sides, we get $\ln y = \ln 3(x^2+1)$, this implies $\frac{1}{y}y' = \ln 3(2x)$. Mutliplying y on both sides, we see $y' = 2x(\ln 3)3^{x^2+1}$.

(a) $2x(\ln 3)3^{x^2+1}$

(b) 3^{2x}

(c) $(\ln 3)3^{2x}$

(d) $(\ln 3)3^{x^2+1}$

(e) $(x^2+1)3^{x^2}$

5.(7pts) Solve for x in the following equation

$$\log_3(3^x + 1) - \log_3(3^x) = 2.$$

(a) $x = \log_3\left(\dfrac{1}{8}\right)$

The equation is equivalent to $\log_3 \frac{3^x+1}{3^x} = 2$, which then becomes $\frac{3^x+1}{3^x} = 3^2$, so we further have $3^x + 1 = 3^{2+x} = 9 \times 3^x$. This implies $8 \times 3^x = 1$ and $3^x = \frac{1}{8}$. Take $\log_3$ on both sides, we have $x = \log_3\left(\frac{1}{8}\right)$.

(a) $x = \log_3\left(\dfrac{1}{8}\right)$

(b) $x = \dfrac{3}{2}$

(c) $x = \dfrac{1}{2}$

(d) $x = 2$

(e) $x = \log_3(8)$

6.(7pts) Find $f'(x)$ if

$$f(x) = \arcsin(e^{2x}).$$

(a) $\dfrac{2e^{2x}}{\sqrt{1-e^{4x}}}$

Because $(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$, by chain rule, we see $(\arcsin(e^{2x}))' = \frac{1}{\sqrt{1-e^{4x}}}(e^{2x})' = \frac{2e^{2x}}{\sqrt{1-e^{4x}}}$.

(a) $\dfrac{2e^{2x}}{\sqrt{1-e^{4x}}}$

(b) $\dfrac{1}{\sqrt{1-e^{4x}}}$

(c) $\dfrac{1}{1+e^{4x}}$

(d) $\dfrac{2e^{2x}}{1+e^{4x}}$

(e) $\dfrac{e^{2x}}{\sqrt{1-e^{2x}}}$

7.(7pts) Determine the following limit:

$$\lim_{x\to\infty} \frac{(\ln x)^2}{x}$$

By L'Hôpital $\lim_{x\to\infty} \frac{(\ln x)^2}{x} = \lim_{x\to\infty} \frac{2\ln x}{x} = \lim_{x\to\infty} \frac{2}{x} = 0.$

(a) 0 (b) $+\infty$ (c) 1 (d) $\frac{1}{2}$ (e) $-\infty$

8.(7pts) Compute

$$\int_0^1 xe^{2x}\, dx.$$

Integration by parts with $u = x$ and $v = e^{2x}/2$, we have

$$\int_0^1 xe^{2x} = \left[xe^{2x}/2\right]_0^1 - \int_0^1 (e^{2x}/2)dx = \left[xe^{2x}/2 - e^{2x}/4\right]_0^1 = \frac{e^2+1}{2}.$$

(a) $\frac{e^2+1}{4}$ (b) $\frac{e^2}{4}$ (c) ∞ (d) -2 (e) $\frac{1}{2}$

9.(7pts) Find $\int_0^{\frac{\pi}{4}} \tan^2 x \ \sec^4 x \ dx$.

Set $u = \tan x$, we have $du = \sec^2 x$ and

$$\int_0^{\pi/4} \tan^2 x \sec^4 x dx = \int_0^{\pi/4} \tan^2 x (1{+}\tan^2 x) \sec^2 x dx = \int_0^1 u^2(1{+}u^2) du = \left[u^3/3 + u^5/5\right]_0^1 = \frac{8}{15}.$$

(a) $\frac{8}{15}$ (b) $\frac{-2}{15}$ (c) $\frac{2}{5}$ (d) $\frac{2}{15}$ (e) 1

10.(7pts) Evaluate the limit

$$\lim_{x\to 0^+} x^x$$

$$\lim_{x\to 0^+} x^x = \lim_{x\to 0^+} e^{x \ln x} = e^0 = 1.$$

(a) 1 (b) e (c) ∞ (d) $-\infty$ (e) $\frac{1}{2}$

11.(7pts) Evaluate the following integral

$$\int \cos(5x)\cos(3x)\; dx.$$

$$\int \cos(5x)\cos(3x)dx = \frac{1}{2}\int \cos(2x) + \cos(8x)dx = \frac{\sin(2x)}{4} + \frac{\sin(8x)}{16} + C$$

(a) $\frac{\sin(2x)}{4} + \frac{\sin(8x)}{16} + C$

(b) $\frac{\sin(2x)}{2} + \frac{\sin(8x)}{2} + C$

(c) $\frac{\sin(5x)\sin(3x)}{15} + C$

(d) $-\frac{\sin(2x)}{2} - \frac{\sin(8x)}{2} + C$

(e) $\cos(2x) + \cos(8x) + C$

12.(7pts) A sample of radioactive material decays to 1/2 of its original amount in one day. Assuming exponential decay, how long would it take for the sample to decay to 1/100 of its original amount?

Let m_0 be its original amount, the function of the exponential decay is given by

$$m(t) = m_0(1/2)^t.$$

Suppose at time t_1, we have $m(t_1) = m_0/100$, then we have

$$t_1 = \log_{1/2}(1/100) = \frac{\ln 1/100}{\ln 1/2} \text{ days.}$$

(a) $\frac{\ln(1/100)}{\ln(1/2)}$ days

(b) $\frac{\ln(1/100)}{\ln(2)}$ days

(c) $\ln(50)$ days

(d) $\frac{\ln(1/100)}{2}$ days

(e) 50 days

Partial Credit

For full credit on partial credit problems, make sure you justify your answers.

13.(10pts) Calculate the integral

$$\int \frac{dx}{\sqrt{x^2+9}}$$

Note: The formula sheet will help you with this problem.

Write your answer in terms of the original variable x and (if needed) replace all composite trigonometric functions (such as $\cos(\sin^{-1}(x/n))$ etc...) by algebraic combinations of x.

Let $x = 3\tan\theta$, so that $dx = 3\sec^2\theta d\theta$. Then the original indefinite integral will become

$$\begin{aligned}\int \frac{dx}{\sqrt{x^2+9}} &= \int \frac{3\sec^2\theta}{\sqrt{9\tan^2\theta+9}}d\theta \\ &= \int \frac{3\sec^2\theta}{3\sec\theta}d\theta \\ &= \int \sec\theta d\theta \\ &= \ln|\sec\theta+\tan\theta| + C.\end{aligned}$$

Since $x = 3\tan\theta$, we have $\theta = \arctan(\frac{x}{3})$. This tells us that

$$\sec\theta = \sec(\arctan(\frac{x}{3})) = \sqrt{1+\tan^2(\arctan(\frac{x}{3}))} = \sqrt{1+\frac{x^2}{9}} = \frac{\sqrt{9+x^2}}{3}.$$

$$\tan\theta = \tan(\arctan(\frac{x}{3})) = \frac{x}{3}.$$

Substitute them back to the indefinite integral we obtained, we learned that

$$\begin{aligned}\int \frac{dx}{\sqrt{x^2+9}} &= \ln|\sec\theta+\tan\theta| + C \\ &= \ln|\frac{\sqrt{9+x^2}+x}{3}| + C.\end{aligned}$$

14.(12pts) Find

$$\int \frac{2x^2+x+2}{x^3+x}\ dx$$

We want to apply partial fraction decomposition. First of all, we have

$$\frac{2x^2+x+2}{x^3+x} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

$$2x^2+x+2 = A(x^2+1)+(Bx+C)x$$

$$2x^2+x+2 = (A+B)x^2+Cx+A.$$

Hence, we have $A=2, B=0, C=1$. Now we consider the following:

$$\int \frac{2x^2+x+2}{x^3+x}\ dx = \int \frac{2}{x} + \frac{1}{x^2+1}\ dx = 2\ln x + \arctan(x) + \text{constant}.$$

True-False.

15.(6pts) Please circle “TRUE” if you think the statement is true, and circle “FALSE” if you think the statement is False.

(a)(1 pt. No Partial credit) $\lim_{x\to\infty} \tan^{-1} x = \infty$. FALSE. The limit of $\tan^{-1} x$ is $\frac{\pi}{2}$.

TRUE FALSE

(b)(1 pt. No Partial credit) $\int \frac{1}{\sqrt{1-x^2}} = \sin^{-1} x + C$. TRUE This is the formula.

TRUE FALSE

(c))(1 pt. No Partial credit) $\frac{5}{x^2-x-6} = \frac{1}{x-3} - \frac{1}{x+2}$. TRUE $\frac{1}{x-3} - \frac{1}{x+2} = \frac{x+2-(x-3)}{(x-3)(x+2)} = \frac{5}{x^2-x-6}$.

TRUE FALSE

(d))(1 pt. No Partial credit) $\frac{d}{dx}e^{2x} = e^{2x}$. FALSE $\frac{d}{dx}e^{2x} = 2e^{2x}$.

TRUE FALSE

(e))(1 pt. No Partial credit) $\sin^{-1}(\sin x) = x$ for any real number x. FALSE the range of $\sin^{-1}$ is $[-\pi/2, \pi/2]$, so $\sin^{-1}(\sin \pi) \neq \pi$

TRUE FALSE

(f))(1 pt. No Partial credit) $\ln\left(\frac{a}{b}\right) = \frac{\ln(a)}{\ln(b)}$ for all positive (> 0) real numbers a and b. FALSE, counterexample: $\ln(\frac{1}{2}) = -\ln 2 \neq 0 = \frac{\ln 1}{\ln 2}$.

TRUE FALSE

16.(2pts) You will be awarded these two points if you write your name in CAPITALS on the front page and you mark your answers on the front page with an X through your answer choice like so: ~~(a)~~ (not an O around your answer choice) .

 Initials: ____________

The following is the list of useful trigonometric formulas:

Note: $\sin^{-1} x$ and $\arcsin(x)$ are different names for the same function and $\tan^{-1} x$ and $\arctan(x)$ are different names for the same function.

$$\sin^2 x + \cos^2 x = 1$$

$$1 + \tan^2 x = \sec^2 x$$

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$$

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

$$\sin 2x = 2 \sin x \cos x$$

$$\sin x \cos y = \frac{1}{2}\big(\sin(x - y) + \sin(x + y)\big)$$

$$\sin x \sin y = \frac{1}{2}\big(\cos(x - y) - \cos(x + y)\big)$$

$$\cos x \cos y = \frac{1}{2}\big(\cos(x - y) + \cos(x + y)\big)$$

$$\int \sec\theta = \ln|\sec\theta + \tan\theta| + C$$

$$\int \csc\theta = \ln|\csc\theta - \cot\theta| + C$$

$$\csc\theta = \frac{1}{\sin\theta}, \quad \cot\theta = \frac{1}{\tan\theta}$$

Initials: ____________

ROUGH WORK